

Semiclassical reduction for magnetic Schrödinger operator with periodic zero-range potentials and applications

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Abstract

The two-dimensional Schrödinger operator with a uniform magnetic field and a periodic zero-range potential is considered. For weak magnetic fields and a weak coupling we reduce the spectral problem to the semiclassical analysis of one-dimensional Harper-like operators. This shows the existence of parts of Cantor structure in the spectrum for special values of the magnetic flux.

Keywords. Schrödinger operator, periodic perturbation, Cantor spectrum, magnetic field, zero-range potential

1 Introduction

The spectral properties of a charged particle in a two-dimensional system submitted to a periodic electric potential and a uniform magnetic field crucially depend on the arithmetic properties of the number θ representing the magnetic flux quanta through the elementary cell of periods, see e.g. [8] for a description of various models. Since the works by Azbel [7] and Hofstadter [29] it is generally believed that for irrational θ the spectrum is a Cantor set, and the graphical presentation of the dependence of the spectrum on θ shows a fractal behavior known as the Hofstadter butterfly. (We note that fractal spectral diagrams has been found recently in the three-dimensional situation as well, see e.g. [9]). After intensive efforts during more than twenty years this was rigorously proved recently (Ten Martini conjecture) for all irrational values of θ for the discrete Hofstadter model, i.e. the discrete magnetic Laplacian admitting a reduction to the almost Mathieu equation, see [3] and references therein.

Only few results are available for other models. Traditionally, semiclassical methods have played an important role in the analysis of the two-dimensional magnetic Schrödinger operators with periodic potentials, see e.g. [10] for a review. In particular, the bottom part of the spectrum for strong magnetic fields can be described up to some extent using the tunneling asymptotics in a very general setting [24]. Concerning a more detailed analysis, it was shown by Helffer and Sjöstrand [25, 26, 27] that the study of some parts of the spectrum for the Schrödinger operator with a magnetic field and a periodic electric potentials reduces to the spectral problem for an operator pencil of one-dimensional quasiperiodic pseudodifferential operators (see below, Section 3); recently this correspondence was extended up to a unitary equivalence [15, 16]. Under some symmetry conditions for the electric potentials, the operator pencil reduces to the study of small perturbation of the continuous analog of the almost-Mathieu operator, which allowed one to carry out a rather detailed iterative analysis for special values of θ . In particular, in several asymptotic regimes a Cantor structure of some part of the spectrum was proved.

In the present paper we are interested in the spectrum of the two-dimensional magnetic Schrödinger operator with periodic *zero-range* potentials (called also point perturbations). Perturbations of such a kind became a rather popular model in quantum mechanics due to a possibility of an analytic investigation whose results are in a good agreement with experiments, see the monographs [2, 13]. Various aspects of the spectral analysis of magnetic Hamiltonians with point perturbations were discussed in numerous works,

see e.g. [4, 5, 14, 17, 18, 19, 22, 32] and references therein. Our principal aim in this paper is to show that the technique of Helffer and Sjöstrand is still applicable and is general enough to handle this non-standard class of operators; such a correspondence between the semiclassical analysis and solvable models seems to appear for the first time. This provides a proof of the existence of Cantor parts in the spectrum in a certain asymptotic situation.

Our starting point will be the construction of [18] for the resolvent of the perturbed operator, which reduces the spectral problem to the study of a family of discrete operators (see Section 2). We then construct a family of pseudodifferential operators (effective Hamiltonians) having the same spectrum as the discrete operators (Section 4). In Section 5 we obtain some estimates for the Green function of the Landau Hamiltonians, which are used to show that in the weak magnetic field limit the effective Hamiltonians obey the conditions needed for the Helffer-Sjöstrand analysis (Section 6 and the main result in Theorem 6.4).

2 Magnetic operator with zero-range potentials

We start with the two-dimensional Schrödinger operator with a uniform magnetic field in the Landau gauge (Landau Hamiltonian),

$$H_h = \left(-i \frac{\partial}{\partial x_1} + hx_2 \right)^2 - \frac{\partial^2}{\partial x_2^2}, \quad h > 0.$$

Recall that the spectrum of H_h consists of the infinitely degenerate eigenvalues $E_n = (2n - 1)h$, $n \in \mathbb{N}$, called the Landau levels. We are going to study periodic zero-range perturbations of H_h supported by the set \mathbb{Z}^2 . Physically, such operators model periodic arrays of identical small impurities. The interactions can be intuitively understood as operators corresponding to the limit for $\varepsilon \rightarrow 0$ of the operators

$$H(\varepsilon) = H_h + \alpha(\varepsilon) \sum_{m \in \mathbb{Z}^2} V_\varepsilon(\cdot - m),$$

for localized potentials V_ε approaching the Dirac δ function and a special choice of the constants $\alpha(\varepsilon)$. An alternative (but essentially equivalent) approach, which is more suitable for the spectral analysis, involves the use of self-adjoint extensions [12]. In this case one considers first the restriction of

H_h to the functions vanishing at all points of \mathbb{Z}^2 (this operator is well defined as all functions in the domain of H_h are continuous), to be denoted by S . By zero-range perturbations of H_h , one means then self-adjoint extensions of S .

We restrict ourselves by considering the standard one-parameter family of operators (self-adjoint extensions) $H_{h,\alpha}$, $\alpha \in \mathbb{R} \setminus \{0\}$, whose domain consists of the functions f having logarithmic singularities at the points of \mathbb{Z}^2 ,

$$f(x) = \frac{a_m(f)}{2\pi} \log \frac{1}{|x-m|} + b_m(f) + o(1) \text{ as } |x-m| = o(1),$$

$$a_m(f), b_m(f) \in \mathbb{C}, \quad m \in \mathbb{Z}^2,$$

and satisfying the “boundary conditions” $a_m(f) + \alpha b_m(f) = 0$, $m \in \mathbb{Z}^2$. The action of the operators in the sense of distributions is given by the following expression (Fermi pseudopotential):

$$H_{h,\alpha}f(x) := H_h f(x) + \alpha \sum_{m \in \mathbb{Z}^2} \delta(x-m) (1 + \log |x-m| \langle x-m, \nabla \rangle) f(x),$$

where δ is the Dirac delta function. The above boundary conditions satisfy the natural assumption of form-locality [31], i.e. the associated bilinear form vanishes on functions with disjoint supports. This helps to avoid some pathological spectral properties appearing for more general boundary conditions [20].

Let us repeat the main constructions of [18] to show how to handle the spectral problem for $H_{h,\alpha}$. Note first that the Green function (the integral kernel of the resolvent) of H_h is known explicitly,

$$G_h(x, y; z) = \frac{1}{4\pi} \exp \left(-\frac{ih(x_1 - y_1)(x_2 + y_2)}{2} - \frac{h(x-y)^2}{4} \right) \\ \times \Gamma \left(\frac{1}{2} - \frac{z}{2h} \right) U \left(\frac{1}{2} - \frac{z}{2h}, 1; \frac{h(x-y)^2}{2} \right), \quad (2.1)$$

where U is the Kummer confluent hypergeometric function. The Green function plays a crucial role in the spectral analysis of the operators $H_{h,\alpha}$. Namely, let us define, for $z \notin \text{spec } H_h$,

$$q(z, h) := \lim_{x \rightarrow y} \left(G_h(x, y; z) - \frac{1}{2\pi} \log \frac{1}{|x-y|} \right) \\ = -\frac{1}{4\pi} \left(\psi \left(\frac{1}{2} - \frac{z}{2h} \right) - \log \frac{h}{2} - 2\psi(1) \right), \quad (2.2)$$

where ψ is the logarithmic derivative of the Γ -function. Note that the limit is independent of the choice of $y \in \mathbb{R}^2$. For $z \notin \text{spec } H_h$ define an operator

$Q(z, h) : \ell^2(\mathbb{Z}^2) \rightarrow \ell^2(\mathbb{Z}^2)$ given in the canonical basis by the matrix

$$Q(m, n; z, h) = \begin{cases} q(z, h), & m = n, \\ G_h(m, n; z), & \text{otherwise.} \end{cases}$$

Then the Green function $G_{h,\alpha}$ of $H_{h,\alpha}$ is given, for $z \notin \text{spec } H_h \cup \text{spec } H_{h,\alpha}$, by

$$G_{h,\alpha}(x, y; z) = G_h(x, y; z) - \alpha \sum_{m, n \in \mathbb{Z}^2} (\alpha Q(z, h) + 1)^{-1}(m, n) G_h(x, m; z) G_h(n, y; z), \quad (2.3)$$

where the series on the right-hand side converges in the strong resolvent sense. An important ingredient of the above formula (2.3) is the relation

$$\text{spec } H_{h,\alpha} \setminus \text{spec } H_h = \{z \notin \text{spec } H_h : 0 \in \text{spec } (Q(z, h) + \alpha^{-1})\}. \quad (2.4)$$

As the set $\text{spec } H_h$ is discrete (the set of the Landau levels), Eq. (2.4) provides an almost complete characterization of the spectrum of $H_{h,\alpha}$, and this is the starting point for the subsequent analysis. Of crucial importance will be also the identity

$$Q(m + k, n + k; z, h) = e^{-ikh_2(m_1 - n_1)} Q(m, n; z, h), \quad m, n, k \in \mathbb{Z}^2, \quad (2.5)$$

which can be verified directly; this expresses the invariance of H_h under the magnetic translations [34].

3 Strong type I operators

In the works [25, 26, 27] a machinery was developed for an iterative semi-classical analysis of a special class of pseudodifferential operators. One was concerned with the non-linear spectral problem (or, in other words, with the spectral problem for an operator pencil). Namely, for a family of self-adjoint operators $A(\mu)$ depending on a real parameter μ , one means by the μ -spectrum $\mu\text{-spec } A(\mu)$ the set of all μ such that $0 \in \text{spec } A(\mu)$. In particular, the μ -spectrum of the family $A(\mu) := A - \mu$ for a self-adjoint operator A is exactly the spectrum of A .

Note that in the above terms the problem (2.4) is exactly to find the z -spectrum for the family $Q(z, h) + \alpha^{-1}$.

Consider a bounded function (symbol) $L : \mathbb{R}^2 \rightarrow \mathbb{C}$, $L = L(x, p)$. By the Weyl quantization procedure one can assign to L an operator \widehat{L}_h in $\mathcal{L}^2(\mathbb{R})$,

$$\widehat{L}_h f(x) = \frac{1}{2\pi h} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{ip(x-y)/h} L\left(\frac{x+y}{2}, p\right) f(y) dp dy. \quad (3.1)$$

Note that L can depend itself on h and other parameters as well. The operator \widehat{L}_h obtained is referred to as the Weyl h -quantization of L (or the h -pseudodifferential operator with the symbol L), and quantum Hamiltonians resulting from symbols periodic in both x and p are often called Harper-like operators. In particular, the symbol $L(x, p) := \cos p + \cos x$ produces the Harper operator on the real line,

$$\widehat{L}_h f(x) = \frac{f(x+h) + f(x-h)}{2} + \cos x f(x). \quad (3.2)$$

In [27], in order to treat the Harper operator and its perturbations occurring in a renormalization procedure, the following notion was introduced, see Definition 3.1 in [27] with $L = P$ and $P_1 = P_2 = I$.

Definition 3.1. *A symbol $L(x, p; \mu, h)$ will be called of strong type I if the following conditions are satisfied for all $h \in (0, h_0)$ with some $h_0 > 0$:*

- (a) *L depends analytically on $\mu \in [-4, 4]$ and takes real values for such μ .*
- (b) *There exists $\varepsilon > 0$ such that*
 - (b1) *$L(x, p; \mu, h)$ is holomorphic in*

$$D_\varepsilon = \left\{ (\mu, x, p) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} : |\mu| \leq 4, |\Im x| + |\Im p| < \frac{1}{\varepsilon} \right\},$$

- (b2) *for $(\mu, x, p) \in D_\varepsilon$, there holds*

$$|L(x, p; \mu, h) - (\cos x + \cos p + \mu)| \leq \varepsilon.$$

- (c) *The following symmetry and periodicity conditions hold:*

$$(c1) \quad L(x, p; \mu, h) = L(-p, x; \mu, h) = L(x, -p; \mu, h),$$

$$(c2) \quad L(x, p; \mu, h) = L(x + 2\pi, p; \mu, h) = L(x, p + 2\pi; \mu, h).$$

By $\varepsilon(L)$ we will denote the infimum of ε for which the above conditions hold.

In [25, 26, 27] a detailed analysis was performed for pseudodifferential operators associated with strong type I symbols. One of the results (appearing as an intermediate stronger result toward the proof of Theorem 0.1 in [27]) was

Theorem 3.2. *Let $L(\mu, h)$ be a strong type I symbol. There exists $\varepsilon_0 > 0$ and $C > 0$ such that if $\varepsilon(L) \leq \varepsilon_0$ and if $(2\pi)^{-1}h$ is an irrational admitting a representation as a continuous fraction*

$$\frac{h}{2\pi} = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}}$$

with integers n_j satisfying $n_j \geq C$, then the μ -spectrum of the associated family of operators $\hat{L}_h(\mu)$ is a zero measure Cantor set.

For operators $H = (-i\nabla + A)^2 + V$ with periodic potentials V ,

$$V(x_1 + 2\pi, x_2) \equiv V(x_1, x_2 + 2\pi) \equiv V(x_1, x_2),$$

and constant magnetic fields, $\text{curl } A = B$, it was shown in several asymptotic regimes that the study of some parts of the spectrum reduces to a non-linear spectral problem of the above type for B^{-1} -pseudodifferential operators with symbols close to $V(x, p)$ for strong magnetic fields, see e.g. [28], or for B -pseudodifferential operators with principal symbols coinciding with the first band function of the zero-field Hamiltonian (Peierls substitution) in the weak magnetic field limit, see [25, Appendix e], [27, p. 117], [28], and earlier contributions by physicists mentioned e.g. in [8]. Hence, strong type I operators appear when considering potentials or first band functions close to $\beta(\cos x_1 + \cos x_2) + \gamma$ for some reals $\beta \neq 0$, γ .

Note that the expression (2.3) is a realization of the Krein resolvent formula for self-adjoint extensions, see e.g. [12], and can be viewed as a kind of the resolvent formula arising in the study of Grushin problems [33], which suggests a certain analogy with the case of regular periodic perturbations studied in [28]. Our aim here is to show that the spectral problem (2.4) for periodic point perturbations can be also reduced in a certain regime (weak magnetic field and weak coupling) to the study of pseudodifferential operators with strong type I symbols.

4 Symbols associated with some discrete operators

It is well known that the spectrum of the operator (3.2) as a set coincides with the spectrum of the discrete magnetic Laplacian acting on $\ell^2(\mathbb{Z}^2)$, see e.g. [25, pages 10–11],

$$C_h f(m, n) = e^{ihn} f(m+1, n) + e^{-ihn} f(m-1, n) + f(m, n-1) + f(m, n+1).$$

The following theorem describes a similar correspondence for more general operators. This is essentially a suitable reformulation of the constructions of Section 6 in [28].

Theorem 4.1. *Let C_h be a bounded linear operator on $\ell^2(\mathbb{Z}^2)$ given by an infinite matrix $(C(p, q))$, $p, q \in \mathbb{Z}^2$, with exponentially decreasing entries, $|C(p, q)| \leq ae^{-b|p-q|}$ for some $a, b > 0$ and all $p, q \in \mathbb{Z}^2$, and satisfying*

$$C(p+k, q+k) = e^{-ihk_2(p_1-q_1)} C(p, q), \quad p, q, k \in \mathbb{Z}^2, \quad (4.1)$$

with some $h > 0$. Then the spectrum of C_h coincides with the spectrum of the Weyl h -quantization of the (h -dependent) symbol T given by

$$T(x, p) = \sum_{m, n \in \mathbb{Z}} c(m, n) e^{-imnh/2} e^{i(mx+np)},$$

where $c(m, n) = C((0, 0), (m, n))$, $m, n \in \mathbb{Z}$.

Proof. Let us start with preliminary constructions. Let $T : \mathbb{R}^2 \rightarrow \mathbb{C}$ be a periodic smooth function, $T(x, p+2\pi) = T(x+2\pi, p) = T(x, p)$. Consider its Fourier expansion,

$$\begin{aligned} T(x, p) &= \sum_{m, n \in \mathbb{Z}} t(m, n) e^{i(mx+np)}, \\ t(m, n) &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} T(x, p) e^{-i(mx+np)} dp dx. \end{aligned} \quad (4.2)$$

We will assume that the coefficients $t(m, n)$ are exponentially decaying, i.e. that

$$|t(m, n)| \leq ae^{-b\sqrt{m^2+n^2}}, \quad \text{for some } a, b > 0.$$

Let us find an expression for the corresponding operator \widehat{T}_h given by (3.1). Let $f \in \mathcal{S}(\mathbb{R})$. Substituting (4.2) into (3.1) we obtain

$$\begin{aligned}\widehat{T}_h f(x) &= \frac{1}{2\pi h} \sum_{m,n \in \mathbb{Z}} t(m,n) \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(np+m(x+y)/2)} e^{ip(x-y)/h} f(y) dp dy \\ &= \frac{1}{2\pi h} \sum_{m,n \in \mathbb{Z}} t(m,n) e^{imx/2} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i(np+my/2)} e^{ip(x-y)/h} f(y) dp dy. \quad (4.3)\end{aligned}$$

Note that for each $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ there holds, in the sense of distributions,

$$\int_{\mathbb{R}} e^{ip(x-y)/h+inp} dp = -2\pi \delta\left(\frac{x-y}{h} + n\right).$$

Hence Eq. (4.3) rewrites as

$$\begin{aligned}\widehat{T}_h f(x) &= \frac{1}{h} \sum_{m,n \in \mathbb{Z}} t(m,n) e^{imx/2} \int_{\mathbb{R}} e^{imy/2} \delta\left(\frac{x-y}{h} + n\right) f(y) dy \\ &= \sum_{m,n \in \mathbb{Z}} t(m,n) e^{imx/2} \left(e^{imy/2} f(y) \right)_{y=x+nh} \\ &= \sum_{m,n \in \mathbb{Z}} t(m,n) e^{imx} e^{imnh/2} f(x+nh), \quad (4.4)\end{aligned}$$

and due to the exponential decay of the coefficients $t(m,n)$ this extends by continuity to the whole of $\mathcal{L}^2(\mathbb{R})$.

Let us return to the initial operator C_h . By the assumption (4.1),

$$C(p, q) = \exp(ihp_2(q_1 - p_1)) c(q - p) \text{ for any } p, q \in \mathbb{Z}^2.$$

Hence

$$C_h f(p) = \sum_{q \in \mathbb{Z}^2} e^{ihp_2(q_1 - p_1)} c(q - p) f(q) = \sum_{q \in \mathbb{Z}^2} e^{ihp_2 q_1} c(q) f(p + q).$$

Therefore, C_h commutes with the shift $f(p_1, p_2) \mapsto f(p_1 + 1, p_2)$, and the Floquet-Bloch theory is applicable.

Let us introduce the functions

$$\mathbb{R} \ni \varphi \mapsto b_n(\varphi) = \sum_{k \in \mathbb{Z}} c(k, n) e^{ik\varphi}, \quad n \in \mathbb{Z}.$$

All these functions are 2π -periodic and analytic in a complex neighborhood of \mathbb{R} . Consider a family (indexed by $\theta \in \mathbb{R}$) of operators $C_h(\theta)$ acting in $\ell^2(\mathbb{Z})$,

$$C_h(\theta) g(m) = \sum_{n \in \mathbb{Z}} b_n(mh + \theta) g(m + n), \quad m \in \mathbb{Z}.$$

The operator-valued function $\theta \mapsto C_h(\theta)$ is obviously continuous in the norm topology. Due to periodicity of the functions b_n one has

$$C_h(\theta) = C_h(\theta + 2\pi) .$$

Therefore, by the Floquet-Bloch theory, one has

$$\text{spec } C_h = \bigcup_{\theta \in [0, 2\pi)} \text{spec } C_h(\theta) . \quad (4.5)$$

Furthermore, for any θ the operators $C_h(\theta)$ and $C_h(\theta + h)$ are unitarily equivalent,

$$C_h(\theta + h) = S C_h(\theta) S^{-1} ,$$

where S is the shift in $\ell^2(\mathbb{Z})$,

$$Sf(n) = f(n + 1) .$$

This implies, cf. (4.5),

$$\text{spec } C_h = \bigcup_{\theta \in [0, h)} \text{spec } C_h(\theta) ,$$

which coincides with the spectrum of the following operator T_h acting in $\mathcal{L}^2(\mathbb{Z} \times [0, h))$:

$$T_h u(m, \theta) = C_h(\theta) u_\theta(m), \quad u_\theta(m) = u(m, \theta), \quad m \in \mathbb{Z}.$$

Consider the map $U : \mathcal{L}^2(\mathbb{R}) \rightarrow \mathcal{L}^2(\mathbb{Z} \times [0, h))$ given by

$$Uf(m, \theta) = f(mh + \theta) , \quad (m, \theta) \in \mathbb{Z} \times [0, h) .$$

One can easily see that U is unitary. The operator $\widehat{T}_h := U^{-1} T_h U$ is then an operator in $\mathcal{L}^2(\mathbb{R})$,

$$\widehat{T}_h f(x) = \sum_{n \in \mathbb{Z}} b_n(x) f(x + nh) = \sum_{m, n \in \mathbb{Z}} c(m, n) e^{imx} f(x + nh).$$

Comparing this expression with (4.4) we arrive at the conclusion. \square

Using Theorem 4.1 and the relations (2.4) and (2.5) one arrives at the following corollary.

Corollary 4.2. *The set $\text{spec } H_{h,\alpha} \setminus \text{spec } H_h$ coincides with the z -spectrum of the family of h -pseudodifferential operators corresponding to the symbols*

$$\begin{aligned} L_\alpha(x, p; z, h) &= \sum_{m,n \in \mathbb{Z}} Q((0,0), (m,n); z) e^{-imnh/2} e^{i(mx+np)} + \alpha^{-1} \\ &= q(z, h) + \alpha^{-1} + \sum_{\substack{m,n \in \mathbb{Z}, \\ |m|+|n| \geq 1}} \lambda(m, n; z, h) e^{i(mx+np)} \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} \lambda(m, n; z, h) &= F_h((0,0), (m,n); z) \\ &= \frac{1}{4\pi} \exp\left(-\frac{h(m^2+n^2)}{4}\right) \Gamma\left(\frac{1}{2} - \frac{z}{2h}\right) U\left(\frac{1}{2} - \frac{z}{2h}, 1; \frac{h(m^2+n^2)}{2}\right), \end{aligned} \quad (4.7)$$

and

$$F_h(x, y; z) = \exp\left(\frac{ih(x_1 - y_1)(x_2 + y_2)}{2}\right) G_h(x, y; z). \quad (4.8)$$

The operators $\widehat{L}_{h,\alpha}$ associated with the above symbols L_α can be viewed as *effective Hamiltonians* of the problem; we emphasize that these are given in an explicit form (the Fourier expansion). We are going to show that after suitable transformations one arrives at the study of strong type I symbols.

5 Estimates for the Green function

In the present section we will establish some estimates involving the Green function, the spectral parameter, and the strength of the magnetic field in the form which will be of importance below. The representation (2.1) works effectively near the Landau levels, see for example [30], while we intend to consider the negative halfline. So we prefer to use the corresponding heat kernel, i.e. the integral kernel of the semigroup e^{-tH_h} , $t > 0$, see e.g. Section 6.2.1.5 in [23],

$$\begin{aligned} P_h(x, y; t) &= \frac{h}{4\pi \sinh ht} \exp\left(-\frac{ih(x_1 - y_1)(x_2 + y_2)}{2}\right) \\ &\quad \times \exp\left(-\frac{h(x - y)^2 \coth ht}{4}\right). \end{aligned} \quad (5.1)$$

To unify the notation, we denote by H_0 the free Laplacian (i.e., the operator H_h with $h = 0$). The above expression (5.1) will be used for $h > 0$, and by

P_0 we denote the heat kernel associated with the free Laplacian,

$$P_0(x, y; t) = \frac{1}{4\pi t} \exp\left(-\frac{(x-y)^2}{4t}\right).$$

For the Green functions we will use the integral representation

$$G_h(x, y; z) = \int_0^\infty e^{zt} P_h(x, y; t) dt, \quad h \geq 0, \quad (5.2)$$

which is valid at least for $\Re z < h$. For the function F_h introduced in (4.8) one has

$$F_h(x, y; z) = \int_0^\infty e^{zt} |P_h(x, y; t)| dt, \quad h > 0. \quad (5.3)$$

Recall that the Green function G_0 of the free Laplacian is known explicitly as well,

$$G_0(x, y; z) = \frac{1}{2\pi} K_0(\sqrt{-z} |x - y|), \quad (5.4)$$

where K_0 is the modified Bessel function of order zero. Here and below we fix the branch of the square root which is a continuation of the usual square root on the positive half-line.

We will also use the integral kernel $G_h^{(2)}(x, y; z)$ of the operator $(H_h - z)^{-2}$, $h \geq 0$, $\Re z < 0$, and the function

$$F_h^{(2)}(x, y; z) = \exp\left(\frac{ih(x_1 - y_1)(x_2 + y_2)}{2}\right) G_h^{(2)}(x, y; z).$$

By the Hilbert resolvent identity, there holds

$$\frac{\partial G_h(x, y; z)}{\partial z} = G_h^{(2)}(x, y; z), \quad (5.5)$$

and, at the same time,

$$\frac{\partial F_h(x, y; z)}{\partial z} = F_h^{(2)}(x, y; z). \quad (5.6)$$

Using the identity $K_0'(w) = -K_1(w)$, see Eq. (9.6.27) in [1], and Eq. (5.4) above, we obtain

$$G_0^{(2)}(x, y; z) = \frac{|x - y|}{4\pi\sqrt{-z}} K_1(\sqrt{-z} |x - y|). \quad (5.7)$$

On the other hand, using

$$(H_h - z)^{-2} = \int_0^\infty t e^{-t(H_h - z)} dt, \quad \Re z < 0, \quad h \geq 0$$

we arrive at

$$G_h^{(2)}(x, y; z) = \int_0^\infty t e^{zt} P_h(x, y; t) dt, \quad \Re z < h, \quad h \geq 0, \quad (5.8)$$

$$F_h^{(2)}(x, y; z) = \int_0^\infty t e^{zt} |P_h(x, y; t)| dt, \quad \Re z < h, \quad h > 0. \quad (5.9)$$

The following two-side estimate will be of importance.

Lemma 5.1. *For $h > 0$ there holds*

$$\exp\left(-ht - \frac{h(x-y)^2}{4}\right) P_0(x, y; t) \leq |P_h(x, y; t)| \leq P_0(x, y; t)$$

for any $t > 0$ and $x, y \in \mathbb{R}^2$.

Proof. The right-hand side inequality is nothing but the Kato (diamagnetic) inequality valid for much more general magnetic fields, see e.g. §2 in [6], so we only need to prove the left-hand side inequality. We will use the elementary estimates

$$\frac{e^t - 1}{t} \geq 1 \text{ and } \frac{t}{1 - e^{-t}} \geq 1 \text{ for } t > 0. \quad (5.10)$$

Using (5.10) one has

$$\frac{h}{\sinh ht} = \frac{e^{-ht}}{t} \frac{2ht}{1 - e^{-2ht}} \geq \frac{e^{-ht}}{t}.$$

At the same time, using (5.10) again, one obtains

$$h \coth ht = h \left(1 + \frac{2}{e^{2ht} - 1}\right) = h + \frac{1}{t} \frac{2ht}{e^{2ht} - 1} \leq h + \frac{1}{t}.$$

Therefore,

$$\begin{aligned} |P_h(x, y; t)| &= \frac{h}{4\pi \sinh ht} \exp\left(-\frac{h(x-y)^2 \coth ht}{4}\right) \\ &\geq \frac{1}{4\pi t} \exp\left(-ht - \frac{h(x-y)^2}{4} - \frac{(x-y)^2}{4t}\right) \\ &= \exp\left(-ht - \frac{h(x-y)^2}{4}\right) P_0(x, y; t). \end{aligned} \quad \square$$

Using (5.2) and (5.8) one arrives at the following corollary.

Corollary 5.2. For $x, y \in \mathbb{R}^2$, $x \neq y$, and real z , $z < h$, one has

$$|G_h(x, y; z)| \geq G_0(x, y; z - h) \exp\left(-\frac{h(x - y)^2}{4}\right)$$

and, for any $x, y \in \mathbb{R}^2$ and real z with $z < h$, there holds

$$|G_h(x, y; z)| \geq G_0^{(2)}(x, y; z - h) \exp\left(-\frac{h(x - y)^2}{4}\right).$$

Lemma 5.3. For $h > 0$ and $\Re z < 0$ there holds

$$\begin{aligned} |F_h(x, y; z) - G_0(x, y; z)| \\ \leq G_0(x, y; \Re z) - G_0(x, y; \Re z - h) \exp\left(-\frac{h(x - y)^2}{4}\right) \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} |F_h^{(2)}(x, y; z) - G_0^{(2)}(x, y; z)| \\ \leq G_0^{(2)}(x, y; \Re z) - G_0^{(2)}(x, y; \Re z - h) \exp\left(-\frac{h(x - y)^2}{4}\right). \end{aligned} \quad (5.12)$$

Proof. Using (5.2) and (5.3) one has

$$\begin{aligned} |F_h(x, y; z) - G_0(x, y; z)| &= \left| \int_0^\infty e^{zt} (|P_h(x, y; t)| - P_0(x, y; t)) dt \right| \\ &\leq \int_0^\infty |e^{zt}| \cdot \left| |P_h(x, y; t)| - P_0(x, y; t) \right| dt. \end{aligned}$$

Estimate $|e^{zt}| = e^{\Re z t}$ and, using Lemma 5.1,

$$\begin{aligned} \left| |P_h(x, y; t)| - P_0(x, y; t) \right| &= P_0(x, y; t) - |P_h(x, y; t)| \\ &\leq P_0(x, y; t) - \exp\left(-ht - \frac{h(x - y)^2}{4}\right) P_0(x, y; t). \end{aligned}$$

We obtain, using (5.2) again,

$$\begin{aligned} |F_h(x, y; z) - G_0(x, y; z)| &\leq \int_0^\infty e^{\Re z t} \left(P_0(x, y; t) - \exp\left(-ht - \frac{h(x - y)^2}{4}\right) P_0(x, y; t) \right) dt \\ &= \int_0^\infty e^{\Re z t} P_0(x, y; t) dt - \exp\left(-\frac{h(x - y)^2}{4}\right) \int_0^\infty e^{(\Re z - h)t} P_0(x, y; t) dt \\ &= G_0(x, y; \Re z) - G_0(x, y; \Re z - h) \exp\left(-\frac{h(x - y)^2}{4}\right). \end{aligned}$$

Eq. (5.12) is obtained in the same way using (5.8) and (5.9). \square

In what follows we need the function

$$q_0(z) := \lim_{x \rightarrow y} \left(G_0(x, y; z) - \frac{1}{2\pi} \log \frac{1}{|x - y|} \right). \quad (5.13)$$

Using Eq. (9.6.13) in [1],

$$\lim_{r \rightarrow 0+} (K_0(wr) + \log r) = -\log w + \log 2 + \psi(1), \quad w \in \mathbb{C} \setminus (-\infty, 0],$$

we obtain

$$q_0(z) = -\frac{\log(-z) - \log 4 - 2\psi(1)}{4\pi}. \quad (5.14)$$

Everywhere we take the principal branch of the logarithm, i.e. the holomorphic extension to $\mathbb{C} \setminus (-\infty, 0]$ of the usual logarithm on $(0, +\infty)$.

Lemma 5.4. *For $h > 0$ and $\Re z < 0$ there holds*

$$|q(z, h) - q_0(z)| \leq \frac{1}{4\pi} \log \frac{\Re z - h}{\Re z}, \quad (5.15)$$

$$\left| \frac{\partial q(z, h)}{\partial z} - \frac{\partial q_0(z)}{\partial z} \right| \leq \frac{h}{4\pi \Re z (\Re z - h)}. \quad (5.16)$$

Proof. Consider first (5.15). Due to the logarithmic on-diagonal singularity of the Green functions one has

$$\begin{aligned} & \lim_{x \rightarrow y} (G_h(x, y; z) - F_h(x, y; z)) \\ &= \lim_{x \rightarrow y} \left(1 - \exp \left(\frac{ih(x_1 - y_1)(x_2 + y_2)}{2} \right) \right) G_h(x, y; z) = 0. \end{aligned}$$

Hence, using (2.2),

$$q(z, h) = \lim_{x \rightarrow y} \left(F_h(x, y; z) - \frac{1}{2\pi} \log \frac{1}{|x - y|} \right). \quad (5.17)$$

Substituting now (5.17) and (5.13) into (5.11) we see

$$\begin{aligned} |q(z, h) - q_0(z)| &= \lim_{x \rightarrow y} \left| F_h(x, y; z) - G_0(x, y; z) \right| \\ &\leq \lim_{x \rightarrow y} \left(G_0(x, y; \Re z) - G_0(x, y; \Re z - h) \exp \left(-\frac{h(x - y)^2}{4} \right) \right) \\ &= \lim_{x \rightarrow y} \left(G_0(x, y; \Re z) - \frac{1}{2\pi} \log \frac{1}{|x - y|} \right) \\ &\quad - \lim_{x \rightarrow y} \left(G_0(x, y; \Re z - h) - \frac{1}{2\pi} \log \frac{1}{|x - y|} \right) \exp \left(-\frac{h(x - y)^2}{4} \right) \\ &\quad + \lim_{x \rightarrow y} \frac{1}{2\pi} \left(1 - \exp \left(-\frac{h(x - y)^2}{4} \right) \right) \log \frac{1}{|x - y|} = q_0(\Re z) - q_0(\Re z - h), \end{aligned}$$

and now one can use the explicit expression (5.14).

As for (5.16), the expressions (2.2) for $q(z, h)$ and (5.13) for $q_0(z)$ together with the analyticity of G_h in z ($h \geq 0$) imply

$$\frac{\partial q(z, h)}{\partial z} = G_h^{(2)}(x, x; z) = F_h^{(2)}(x, x; z), \quad \frac{\partial q_0(z)}{\partial z} = G_0^{(2)}(x, x; z),$$

where $x \in \mathbb{R}^2$ is arbitrary. Therefore, using (5.12), one obtains

$$\left| \frac{\partial q(z, h)}{\partial z} - \frac{\partial q_0(z)}{\partial z} \right| \leq G_0^{(2)}(x, x; \Re z) - G_0^{(2)}(x, x; \Re z - h) = \frac{\partial q_0(\lambda)}{\partial \lambda} \Big|_{\lambda=\Re z-h}^{\lambda=\Re z},$$

and, by (5.14), one has

$$\frac{\partial q_0(\lambda)}{\partial \lambda} = -\frac{1}{4\pi\lambda}$$

which finishes the proof. \square

Below we will use intensively the following well-known asymptotics, Eq. (9.7.2) in [1]:

$$K_0(w) = \sqrt{\frac{\pi}{2w}} e^{-w} (1 + O(w^{-1})), \quad |w| \rightarrow \infty, \quad |\arg w| \leq \frac{\pi}{4}. \quad (5.18)$$

Let us prove first several estimates from above for the Green function and its derivative.

Lemma 5.5. *For any $d, \delta > 0$ there exists $E > 0$ such that for $|x - y| > d$, $\Re z < -E$, and $h \geq 0$ there holds*

$$|G_h(x, y; z)| \leq \frac{1 + \delta}{2\pi} \sqrt[4]{\frac{\pi^2}{-4\Re z |x - y|^2}} e^{-\sqrt{-\Re z} |x - y|}.$$

Proof. Using (5.2) and the right-hand side inequality of Lemma 5.1, for $\Re z < 0$ we obtain

$$\begin{aligned} |G_h(x, y; z)| &= \left| \int_0^\infty e^{zt} P_h(x, y; t) dt \right| \\ &\leq \int_0^\infty |e^{zt}| \cdot |P_h(x, y; t)| dt \leq \int_0^\infty e^{\Re z t} P_0(x, y; t) dt = G_0(x, y; \Re z). \end{aligned}$$

Substituting here (5.4) one obtains

$$|G_h(x, y; z)| \leq \frac{1}{2\pi} K_0(\sqrt{-\Re z} |x - y|),$$

and the asymptotics (5.18) gives the result. \square

Lemma 5.6. *For any $d, \delta > 0$ there exists $E > 0$ such that for $|x - y| > d$, $\Re z < -E$, and $h \geq 0$ there holds*

$$\left| \frac{\partial G_h(x, y; z)}{\partial z} \right| \leq \frac{(1 + \delta)}{4\pi} \sqrt[4]{\frac{\pi^2 |x - y|^2}{-4(\Re z)^3}} \exp(-\sqrt{-\Re z} |x - y|).$$

Proof. Using Eqs. (5.5), (5.8), and the right-hand side inequality of Lemma 5.1, for $\Re z < 0$ we obtain

$$\begin{aligned} \left| \frac{\partial G_h(x, y; z)}{\partial z} \right| &= |G_h^{(2)}(x, y; z)| \leq \int_0^\infty t |e^{zt}| \cdot |P_h(x, y; t)| dt \\ &\leq \int_0^\infty t e^{\Re z t} P_0(x, y; t) dt = G_0^{(2)}(x, y; \Re z). \end{aligned}$$

Hence, using (5.7) we arrive at

$$\left| \frac{\partial G_h(x, y; z)}{\partial z} \right| \leq \frac{|x - y|}{4\pi \sqrt{-\Re z}} K_1(\sqrt{-\Re z} |x - y|),$$

and it is sufficient to use the asymptotics

$$K_1(w) = \sqrt{\frac{\pi}{2w}} e^{-w} (1 + O(w^{-1})), \quad |w| \rightarrow \infty, \quad |\arg w| \leq \frac{\pi}{4},$$

see Eq. (9.7.2) in [1]. □

Let us pass to estimates from below, which are of crucial importance.

Lemma 5.7. *For any $x, y \in \mathbb{R}^2$, $x \neq y$, and any $c \in (0, 1)$ there exist $E, \gamma, h_0 > 0$ (which depend on x, y, c) such that*

$$|G_h(x, y; z)| \geq \frac{1 - c}{2\pi} \sqrt[4]{\frac{\pi^2}{-4\Re z |x - y|^2}} e^{-\sqrt{-\Re z} |x - y|}$$

for $\Re z < -E$, $|\Im z| < \gamma$, and $h \in (0, h_0)$.

Proof. By Lemma 5.3,

$$\begin{aligned} |G_h(x, y; z)| &= |F_h(x, y; z)| \\ &\geq |G_0(x, y; z)| - |F_h(x, y; z) - G_0(x, y; z)| \\ &\geq |G_0(x, y; z)| + G_0(x, y; \Re z - h) \exp\left(-\frac{h(x - y)^2}{4}\right) - G_0(x, y; \Re z) \\ &= |G_0(x, y; z)| + G_0(x, y; \Re z - h) \left(\exp\left(-\frac{h(x - y)^2}{4}\right) - 1 \right) \\ &\quad + (G_0(x, y; \Re z - h) - G_0(x, y; \Re z)). \end{aligned} \tag{5.19}$$

Due to (5.18) and Lemma 5.5, for any $\delta \in (0, 1)$ one can choose $E(\delta, x, y) > 0$ such that

$$\begin{aligned} \frac{1-\delta}{2\pi} \left| \sqrt[4]{\frac{\pi^2}{-4z|x-y|^2}} e^{-\sqrt{-z}|x-y|} \right| &\leq |G_0(x, y; z)| \\ &\leq \frac{1+\delta}{2\pi} \sqrt[4]{\frac{\pi^2}{-4\Re z|x-y|^2}} e^{-\sqrt{-\Re z}|x-y|}, \quad \Re z < -E(\delta, x, y). \end{aligned} \quad (5.20)$$

The constant δ will be chosen later.

One can find $h_0 = h_0(\delta, x, y)$ such that

$$\left| \exp\left(-\frac{h_0(x-y)^2}{4}\right) - 1 \right| \leq \delta, \quad (5.21)$$

then, by the right-hand side inequality in (5.20),

$$\begin{aligned} &\left| G_0(x, y; \Re z - h) \left(\exp\left(-\frac{h(x-y)^2}{4}\right) - 1 \right) \right| \\ &\leq \frac{\delta(1+\delta)}{2\pi} \sqrt[4]{\frac{\pi^2}{4(h-\Re z)|x-y|^2}} e^{-\sqrt{h-\Re z}|x-y|} \\ &\leq \frac{\delta(1+\delta)}{2\pi} \sqrt[4]{\frac{\pi^2}{-4\Re z|x-y|^2}} e^{-\sqrt{-\Re z}|x-y|} \\ &\quad \text{for } \Re z < -E(\delta, x, y), \quad h \in (0, h_0(\delta, x, y)). \end{aligned} \quad (5.22)$$

At the same time one has

$$\begin{aligned} \left| \frac{\exp(-\sqrt{-z}|x-y|)}{\sqrt[4]{-z}} \right| &= \frac{\exp(-\Re \sqrt{-z}|x-y|)}{\sqrt[4]{|z|}} \\ &\geq \frac{\exp(-|\sqrt{-z}||x-y|)}{\sqrt[4]{|z|}} = \frac{\exp(-\sqrt{|z|}|x-y|)}{\sqrt[4]{|z|}} \\ &\geq \frac{1}{\sqrt[4]{|\Re z| + |\Im z|}} \exp(-\sqrt{|\Re z| + |\Im z|}|x-y|), \end{aligned}$$

hence for $|\Im z| < \gamma$ one can estimate

$$\left| \frac{1}{\sqrt[4]{-z}} \exp(-\sqrt{-z}|x-y|) \right| \geq \frac{1}{\sqrt[4]{\gamma + |\Re z|}} \exp(-\sqrt{\gamma + |\Re z|}|x-y|). \quad (5.23)$$

For $t > E > 0$ and $\gamma > 0$ one can write

$$\sqrt{t+\gamma} - \sqrt{t} = \frac{\gamma}{\sqrt{t+\gamma} + \sqrt{t}} \leq \frac{\gamma}{2\sqrt{t}} \leq \frac{\gamma}{2\sqrt{E}}.$$

Hence, for any $\alpha, \gamma > 0$ and $t > E > 0$, one has

$$\exp(-\alpha\sqrt{t}) \leq \exp\left(\frac{\alpha\gamma}{2\sqrt{E}}\right) \exp(-\alpha\sqrt{t+\gamma}). \quad (5.24)$$

Analogously one has for $t > E > 0$ and $\gamma > 0$,

$$\sqrt[4]{\gamma+t} = \sqrt[4]{1+\frac{\gamma}{t}} \sqrt[4]{t} \leq \left(1+\frac{\gamma}{t}\right) \sqrt[4]{t} \leq \left(1+\frac{\gamma}{E}\right) \sqrt[4]{t}. \quad (5.25)$$

Hence, assuming that $\gamma = \gamma(\delta, x, y)$ satisfies

$$\exp \frac{\gamma|x-y|}{2\sqrt{E}} \leq 1 + \delta, \quad \frac{\gamma}{E} \leq \delta,$$

and substituting (5.23), (5.24), and (5.25) into the left-hand side inequality in (5.20) one arrives at

$$\begin{aligned} |G_0(x, y; z)| &\geq \frac{1-\delta}{2\pi(1+\delta)^2} \sqrt[4]{\frac{\pi^2}{-4\Re z |x-y|^2}} e^{-\sqrt{-\Re z}|x-y|} \\ &\quad \text{for } \Re z < -E(\delta, x, y) \text{ and } |\Im z| < \gamma(\delta, x, y). \end{aligned} \quad (5.26)$$

The estimates (5.20) give also, for $\Re z < -E$,

$$\begin{aligned} &|G_0(x, y; \Re z - h) - G_0(x, y; \Re z)| \\ &\leq \frac{1+\delta}{2\pi} \sqrt[4]{\frac{\pi^2}{-4\Re z |x-y|^2}} e^{-\sqrt{-\Re z}|x-y|} \\ &\quad - \frac{1-\delta}{2\pi} \sqrt[4]{\frac{\pi^2}{4(h-\Re z) |x-y|^2}} e^{-\sqrt{h-\Re z}|x-y|} \\ &\leq \frac{1}{2\pi} \left(\sqrt[4]{\frac{\pi^2}{-4\Re z |x-y|^2}} e^{-\sqrt{-\Re z}|x-y|} - \sqrt[4]{\frac{\pi^2}{4(h-\Re z) |x-y|^2}} e^{-\sqrt{h-\Re z}|x-y|} \right) \\ &\quad + \frac{\delta}{\pi} \sqrt[4]{\frac{\pi^2}{-4\Re z |x-y|^2}} e^{-\sqrt{-\Re z}|x-y|}. \end{aligned}$$

Expand

$$\begin{aligned}
& \sqrt[4]{\frac{\pi^2}{-4\Re z |x-y|^2}} e^{-\sqrt{-\Re z}|x-y|} - \sqrt[4]{\frac{\pi^2}{4(h-\Re z) |x-y|^2}} e^{-\sqrt{h-\Re z}|x-y|} \\
&= \sqrt[4]{\frac{\pi^2}{-4\Re z |x-y|^2}} \left(e^{-\sqrt{-\Re z}|x-y|} - e^{-\sqrt{h-\Re z}|x-y|} \right) \\
&+ \left(\sqrt[4]{\frac{\pi^2}{-4\Re z |x-y|^2}} - \sqrt[4]{\frac{\pi^2}{4(h-\Re z) |x-y|^2}} \right) e^{-\sqrt{h-\Re z}|x-y|}.
\end{aligned}$$

Using (5.24) we estimate, for $h \in (0, h_0)$ and $\Re z < -E$,

$$\begin{aligned}
& \exp(-\sqrt{-\Re z}|x-y|) - \exp(-\sqrt{h-\Re z}|x-y|) \\
&\leq \left(\exp \frac{h|x-y|}{2\sqrt{E}} - 1 \right) \exp(-\sqrt{h-\Re z}|x-y|) \\
&\leq \left(\exp \frac{h_0|x-y|}{2\sqrt{E}} - 1 \right) \exp(-\sqrt{-\Re z}|x-y|).
\end{aligned}$$

At the same time one has, for $h \in (0, h_0)$ and $\Re z < -E$,

$$\begin{aligned}
& \sqrt[4]{\frac{\pi^2}{-4\Re z |x-y|^2}} - \sqrt[4]{\frac{\pi^2}{4(h-\Re z) |x-y|^2}} \\
&= \left(\sqrt[4]{\frac{h-\Re z}{-\Re z}} - 1 \right) \sqrt[4]{\frac{\pi^2}{4(h-\Re z) |x-y|^2}} \\
&\leq \frac{h}{|\Re z|} \sqrt[4]{\frac{\pi^2}{4|\Re z| |x-y|^2}} \leq \frac{h_0}{E} \sqrt[4]{\frac{\pi^2}{4|\Re z| |x-y|^2}}.
\end{aligned}$$

Let us choose $h_0 = h_0(\delta, x, y)$ in such a way that (5.21) still holds and that

$$\exp\left(\frac{h_0|x-y|}{2\sqrt{E}}\right) - 1 < \delta, \quad \frac{h_0}{E} < \delta,$$

then

$$\begin{aligned}
& |G_0(x, y; \Re z - h) - G_0(x, y; \Re z)| \leq \frac{2\delta}{\pi} \sqrt[4]{\frac{\pi^2}{-4\Re z |x-y|^2}} e^{-\sqrt{-\Re z}|x-y|} \\
&\text{for } \Re z < -E(\delta, x, y), \quad |\Im z| < \gamma(\delta, x, y), \quad h \in (0, h_0(\delta, x, y)). \quad (5.27)
\end{aligned}$$

Substituting now the estimates (5.22), (5.26), and (5.27) into (5.19), one shows that

$$|G_h(x, y; z)| \geq \frac{1}{2\pi} \left(\frac{1-\delta}{(1+\delta)^2} - \delta(1+\delta) - 4\delta \right) \sqrt[4]{\frac{\pi^2}{-4\Re z |x-y|^2}} e^{-\sqrt{-\Re z} |x-y|}$$

for $\Re z < -E(\delta, x, y)$, $|\Im z| < \gamma(\delta, x, y)$, and $h \in (0, h_0(\delta, x, y))$, which yields the result due to the arbitrariness in the choice of δ . \square

6 Estimates for the effective Hamiltonians

In this section, we are going to analyze the symbols L_α from Corollary 4.2. One can rewrite (4.6) as follows:

$$\begin{aligned} L_\alpha(x, p; z, h) &= q(z, h) + \alpha^{-1} + a(z, h)(\cos x + \cos p) + W(x, p; z, h), \\ a(z, h) &:= 2\lambda(1, 0; z, h), \\ W(x, p; z, h) &:= \sum_{\substack{m, n \in \mathbb{Z}, \\ |m|+|n| \geq 2}} \lambda(m, n; z, h) e^{i(mx+np)}. \end{aligned}$$

First note that due to the integral representation, Eq. (13.2.5) in [1],

$$\Gamma(a)U(a, c; x) = \int_0^\infty e^{-xt} t^{a-1} (1+t)^{c-a-1} dt, \quad \Re a > 0,$$

and to the formula (2.1) for the Green function one has

$$\begin{aligned} G_h(x, y; z) &= \frac{1}{4\pi} \exp \left(-\frac{ih(x_1 - y_1)(x_2 + y_2)}{2} - \frac{h(x - y)^2}{4} \right) \\ &\quad \times \int_0^\infty e^{-\frac{h(x-y)^2 t}{2}} \left(\frac{t}{t+1} \right)^{\frac{h-z}{2h}} \frac{dt}{t}. \end{aligned}$$

Therefore, by (4.7), there exists $C = C(h) > 0$ such that for $\Re z < 0$ one has

$$|\lambda(m, n; z, h)| \leq C \exp \left(-\frac{h(m^2 + n^2)}{4} \right) \text{ for all } m \text{ and } n, |m| + |n| > 0.$$

Hence L_α is an analytic function of $(x, p) \in \mathbb{C}^2$.

Note that due to Corollary 5.2 and (5.4) one has

$$|a(z, h)| \geq \frac{e^{-h/4}}{2\pi} K_0(\sqrt{h-z}), \quad h > 0, \quad z \in (-\infty, h).$$

As the function $K_0(w)$ is strictly positive for positive w , see e.g. Section 9.6.1 in [1], one can conclude that $a(z, h)$ is non-zero for all positive h and real negative z , and Corollary 4.2 gives

Lemma 6.1. *The negative spectrum of $H_{h,\alpha}$ coincides with the negative z -spectrum of a family of h -pseudodifferential operators associated with the symbols*

$$\begin{aligned} M_\alpha(x, p; z, h) &= \frac{1}{a(z, h)} L_\alpha(x, p; z, h) \\ &= m_\alpha(z, h) + \cos x + \cos p + N(x, p; z, h) \end{aligned} \quad (6.1)$$

with

$$\begin{aligned} m_\alpha(z, h) &= \frac{q(z, h) + \alpha^{-1}}{a(z, h)}, \\ N(x, p; z, h) &= \sum_{\substack{m, n \in \mathbb{Z}, \\ |m| + |n| \geq 2}} \frac{\lambda(m, n; z, h)}{a(z, h)} e^{i(mx + np)}. \end{aligned} \quad (6.2)$$

Lemma 6.2. *Take $h_0 > 0$ and $R > 0$. Then one can find $E_0 > 0$ with the following property: for any $E > E_0$ there exists $\alpha_E > 0$ such that for $\alpha \in (0, \alpha_E)$, $h \in (0, h_0)$, $\Re z < -E_0$ the condition $|q(z, h) + \alpha^{-1}| \leq R|a(z, h)|$ guarantees that $\Re z < -E$ and $|\Im z| < 4h_0$.*

Proof. Take an arbitrary $E' > 0$. Due to Lemma 5.5, there exists C' such that, for $\Re z < -E'$ and $h > 0$, we have the estimate

$$|a(z, h)| \leq C' \exp(-\sqrt{-\Re z}).$$

Therefore, the condition

$$|q(z, h) + \alpha^{-1}| \leq R|a(z, h)|$$

implies

$$|q(z, h) + \alpha^{-1}| \leq C \exp(-\sqrt{-\Re z}), \quad C = C'R.$$

Using the triangular inequality,

$$|q_0(z) + \alpha^{-1}| \leq |q(z, h) + \alpha^{-1}| + |q(z, h) - q_0(z)|,$$

and the estimate (5.15) we obtain

$$|q_0(z) + \alpha^{-1}| \leq \frac{1}{4\pi} \log \frac{\Re z - h}{\Re z} + C e^{-\sqrt{-\Re z}} \leq \frac{h}{4\pi|\Re z|} + C e^{-\sqrt{|\Re z|}}.$$

Substituting here the explicit expression (5.14) we obtain

$$\left| -\frac{\log |z| + i \arg(-z) - \log 4 - 2\psi(1)}{4\pi} + \alpha^{-1} \right| \leq \frac{h}{4\pi|\Re z|} + Ce^{-\sqrt{|\Re z|}}.$$

In particular, considering the real and the imaginary parts separately,

$$\left| -\frac{\log |z| - \log 4 - 2\psi(1)}{4\pi} + \alpha^{-1} \right| \leq \frac{h}{4\pi|\Re z|} + Ce^{-\sqrt{-\Re z}}, \quad (6.3)$$

$$\frac{|\arg(-z)|}{4\pi} \leq \frac{h}{4\pi|\Re z|} + Ce^{-\sqrt{-\Re z}}. \quad (6.4)$$

Using (6.4), let us choose $E_0 > E'$ such that

$$\begin{aligned} |\arg(-z)| &\leq \frac{\pi}{4}, \quad \Re z < -E_0, \quad h \in (0, h_0), \\ \log 2E_0 - \log 4 - 2\psi(1) &> 0. \end{aligned}$$

Then, under the same conditions on z and h , one has

$$|\Re z| \leq |z| \leq 2|\Re z|,$$

and from (6.3) one can get

$$\alpha^{-1} \leq \frac{h}{4\pi|\Re z|} + Ce^{-\sqrt{-\Re z}} + \frac{\log 2|\Re z| - \log 4 - 2\psi(1)}{4\pi}. \quad (6.5)$$

With

$$B := \sup \left\{ \frac{h}{4\pi|\Re z|} + Ce^{-\sqrt{-\Re z}} - \frac{2\psi(1) + \log 4}{4\pi} : \Re z < -E_0, h \in (0, h_0) \right\}$$

the estimate (6.5) gives

$$\alpha^{-1} \leq B + \frac{\log 2|\Re z|}{4\pi}. \quad (6.6)$$

Therefore, for any $E > E_0$ and $\alpha \in (0, \alpha_E)$ with

$$\alpha_E = \frac{4\pi}{4\pi B + \log 2E}$$

the condition (6.6) gives $\Re z < -E$.

Using (6.4) again, one can (if necessary) increase E_0 to obtain

$$|\arg(-z)| \leq \frac{2h_0}{|\Re z|}, \quad \Re z < -E_0,$$

and

$$\tan t \leq 2t \text{ for } t \in \left(0, \frac{2h_0}{E_0}\right),$$

then one has

$$|\Im z| = |\tan \arg(-z)| \cdot |\Re z| < 4h_0$$

for $\Re z < -E_0$. □

Lemma 6.3. *Take $h_0 > 0$ and $R > 0$ and let E_0 be the corresponding constant from Lemma 6.2. There exists $\alpha_0 > 0$ such that, for any $h \in (0, h_0)$ and $\alpha \in (0, \alpha_0)$, the equation*

$$q(z, h) + \alpha^{-1} = \mu a(z, h) \tag{6.7}$$

with respect to z , $\Re z < -E_0$, has a unique solution $z = \zeta_\alpha(\mu, h)$ holomorphic in μ for complex μ with $|\mu| < R$, and this solution is real for real μ .

Proof. For complex β and $\alpha > 0$ denote by $z(\beta, \alpha)$ the solution z to the equation

$$q_0(z) + \alpha^{-1} = \beta.$$

Due to the explicit formula (5.13) for q_0 one has an explicit expression

$$z(\beta, \alpha) = -\exp\left(\log 4 + 2\psi(1) + \frac{4\pi}{\alpha} - 4\pi\beta\right); \tag{6.8}$$

in particular,

$$\Re z(\beta, \alpha) = -\exp\left(\log 4 + 2\psi(1) + \frac{4\pi}{\alpha} - 4\pi\Re\beta\right) \cos(4\pi\Im\beta).$$

Now note that if $\beta = \beta(\alpha, \mu, h)$ satisfies

$$\beta = q_0(z(\beta, \alpha)) - q(z(\beta, \alpha), h) + \mu a(z(\beta, \alpha), h),$$

then $\zeta_\alpha(\mu, h) := z(\beta(\alpha, \mu, h), \alpha)$ is a requested solution for (6.7).

To show the existence of a unique $\beta(\alpha, \mu, h)$ we use the fixed point theorem. We are going to analyze the map Ψ ,

$$\beta \mapsto \Psi(\beta, \alpha, \mu, h) := q_0(z(\beta, \alpha)) - q(z(\beta, \alpha), h) + \mu a(z(\beta, \alpha), h),$$

and to show that it is a contraction for $|\beta| \leq 1/16$ if α is sufficiently small. Namely, we will prove that one can choose α_0 in such a way that

$$|\Psi(\beta, \alpha, \mu, h)| \leq \frac{1}{32} \text{ for } |\beta| \leq \frac{1}{16}, \alpha \in (0, \alpha_0), |\mu| < R, h \in (0, h_0), \tag{6.9}$$

and that

$$\left| \Psi(\beta_1, \alpha, \mu, h) - \Psi(\beta_2, \alpha, \mu, h) \right| \leq \frac{|\beta_1 - \beta_2|}{2}$$

$$\text{for } |\beta_1|, |\beta_2| \leq \frac{1}{16}, \alpha \in (0, \alpha_0), |\mu| < R, h \in (0, h_0), \quad (6.10)$$

then, by the fixed point theorem, the equation $\beta = \Psi(\beta, \alpha, \mu, h)$ has a unique solution with $|\beta| \leq 1/16$ depending holomorphically on μ , which can be obtained through iteration $\beta_{n+1} = \Psi(\beta_n, \alpha, \mu, h)$ with, say, $\beta_0 = 0$, and taking the limit $\beta = \lim \beta_n$. If μ is real, then all β_n are real as well, hence so is β .

Note that for $|\beta| \leq 1/16$ one has

$$\Re z(\beta, \alpha) \leq -\frac{1}{\sqrt{2}} \exp \left(\log 4 + 2\psi(1) + \frac{4\pi}{\alpha} - 4\pi \Re \beta \right). \quad (6.11)$$

To show (6.9) we note that, by Lemma 5.4 and Lemma 5.5 there exists $E_1 > 0$ such that for $\Re z(\beta, \alpha) < -E_1$ and $|\mu| < R$ one has

$$|\Psi(\beta, \alpha, \mu, h)| \leq \frac{h}{4\pi |\Re z(\beta, \alpha)|} + \frac{R}{|\Re z(\beta, \alpha)|^{1/4}} \exp \left(-\sqrt{|\Re z(\beta, \alpha)|} \right).$$

Therefore, there exists $E_2 > E_1$ such that

$$|\Psi(\beta, \alpha, \mu, h)| \leq \frac{1}{32} \text{ for } h \in (0, h_0), |\mu| < R, \Re z(\beta, \alpha) < -E_2,$$

and it remains to use (6.11) to choose $\alpha_1 > 0$ in such a way that $\Re z(\beta, \alpha) < -E_2$ for $\alpha \in (0, \alpha_1)$ and $|\beta| \leq 1/16$.

Let us prove now the estimate (6.10). One has

$$\left| \Psi(\beta_1, \alpha, \mu, h) - \Psi(\beta_2, \alpha, \mu, h) \right| \leq \Phi(\alpha, \mu, h) |\beta_1 - \beta_2|,$$

where

$$\Phi(\alpha, \mu, h) := \sup_{|\beta| \leq 1/16} \left| \frac{\partial \Psi(\beta, \alpha, \mu, h)}{\partial \beta} \right|.$$

Hence it is sufficient to show that

$$\left| \frac{\partial \Psi(\beta, \alpha, \mu, h)}{\partial \beta} \right| \leq 1/2 \quad (6.12)$$

for $|\beta| \leq 1/16$ and small α .

We note first that, by (6.8) and (6.11), for $|\beta| \leq 1/16$ there exist $B, C > 0$ such that

$$\left| \frac{\partial z(\beta, \alpha)}{\partial \beta} \right| \leq B \exp \frac{4\pi}{\alpha} \leq C |\Re z(\beta, \alpha)|. \quad (6.13)$$

By Lemma 5.4 and Lemma 5.6, there exists $E_3 > 0$ such that for $\Re z(\beta, \alpha) < -E_3$ one has

$$\begin{aligned} \left| \frac{\partial \Psi(\beta, \alpha, \mu, h)}{\partial \beta} \right| &\leq \left| \frac{\partial z(\beta, \alpha)}{\partial \beta} \right| \left(\frac{h}{4\pi |\Re z(\beta, \alpha)|^2} + \frac{R}{|\Re z(\beta, \alpha)|^{3/4}} e^{-\sqrt{|\Re z(\beta, \alpha)|}} \right) \\ &\leq \frac{Ch}{4\pi |\Re z(\beta, \alpha)|} + CR |\Re z(\beta, \alpha)|^{1/4} e^{-\sqrt{|\Re z(\beta, \alpha)|}}, \end{aligned}$$

where the last estimate is due to (6.13). Hence one can find $E_4 > E_3$ such that (6.12) holds provided $\Re z(\beta, \alpha) < -E_4$ for $|\beta| \leq 1/16$ and $h \in (0, h_0)$, and one can again pick $\alpha_2 \in (0, \alpha_1)$ in such a way that $\Re z(\beta, \alpha) < -E_4$ for $|\beta| \leq 1/16$ for $\alpha \in (0, \alpha_2)$ using the estimate (6.11).

Now it remains to show that for sufficiently small α any solution z to (6.7) really satisfies $|q_0(z) + \alpha^{-1}| \leq 1/16$; this will mean that the above solution ζ_α is unique. For any such solution z one has $|q(z, h) + \alpha^{-1}| \leq R|a(z, h)|$. Combining this with the triangular inequality and the estimate of Lemma 5.4 one arrives at

$$|q_0(z) + \alpha^{-1}| \leq |q(z, h) + \alpha^{-1}| + |q(z, h) - q_0(z)| \leq R|a(z, h)| + \frac{h}{4\pi |\Re z|}.$$

By Lemma 6.2, for any $E > E_0$ we can find $\alpha_E \in (0, \alpha_2)$ such that for $\alpha \in (0, \alpha_E)$ and $h \in (0, h_0)$ the solution satisfies $\Re z < -E$. Take E sufficiently large to have

$$R|a(z, h)| + \frac{h_0}{4\pi |\Re z|} \leq 1/16 \text{ for } \Re z < -E,$$

which is possible due to the estimate of Lemma 5.6 for $a(z, h)$. Denoting the corresponding value of α_E by α_0 , this completes the proof. \square

Below we will need the fact that $a(z, h)$ does not vanish in a certain tubular complex domain, and not only on the negative halfline. Namely, it follows from Lemma 5.7 that one can find $h_* > 0$ and $E_* > 0$ such that :

$$a(z, h) \neq 0, \text{ if } \Re z < -E_*, |\Im z| < 4h_* \text{ and } h \in (0, h_*).$$

Note that in this domain the equation (6.7) reads simply as $m_\alpha(z, h) = \mu$.

The following theorem summarizes all previous considerations and shows the applicability of the Helffer-Sjöstrand methods [25, 26, 27].

Theorem 6.4. *For any $E_0 > 0$ there exist $\alpha_0 > 0$ and $h_0 > 0$ such that for $\alpha \in (0, \alpha_0)$ and $h \in (0, h_0)$ one has*

$$\text{spec } H_{h,\alpha} \cap (-\infty, -E_0) = \{z < -E_0 : m_\alpha(z, h) \in \mu\text{-spec } \widehat{P}_{h,\alpha}(\mu)\},$$

where m_α is given in (6.2), $\widehat{P}_{h,\alpha}$ is a pseudodifferential operator with a strong type I symbol $P_\alpha(x, p; \mu, h)$ given by Eq. (6.14) below, and $\lim_{\alpha \rightarrow 0} \varepsilon(P_\alpha) = 0$.

Proof. By Corollary 4.2 and Eq. (6.1), the negative spectrum of $H_{h,\alpha}$ coincides with the z -spectrum for the symbols $M_\alpha(x, p; z, h)$. Note that the symmetry conditions (c) in Definition 3.1 are satisfied due to the specific form of the Fourier coefficients, see (4.7), so we will be concerned below with the conditions (a) and (b) of Definition 3.1.

Due to Lemma 5.5 and the estimate (5.15), for any segment $[a, b] \subset (-\infty, 0)$ there exists a constant $C > 0$ such that $\|Q(z, h)\| \leq C$ for $z \in [a, b]$ and any h . By (2.4), this means that $H_{h,\alpha}$ has no spectrum in $[a, b]$ for $|\alpha| < 1/C$. Therefore, we can assume without loss of generality that E_0 satisfies the assumptions of Lemma 6.2 for $R = 5$ and $h_0 = h_*$, and that $E_0 \geq E_*$.

Using Lemma 6.3 one can choose $\alpha_1 > 0$ such that for $h \in (0, h_*)$ and $\alpha \in (0, \alpha_1)$ the equation $m_\alpha(z, h) = \mu$ with respect to z with $\Re z < -E_0$ has a unique solution $z = \zeta_\alpha(\mu, h)$ for all μ with $|\mu| \leq 4$, and this solution is a holomorphic function of μ and is real for real μ .

Let us consider the symbols

$$P_\alpha(x, p; \mu, h) := M_\alpha(x, p; \zeta_\alpha(\mu, h), h). \quad (6.14)$$

Clearly,

$$P_\alpha(x, p; \mu, h) := \mu + \cos x + \cos p + T_\alpha(x, p; \mu, h),$$

where

$$T_\alpha(x, p; \mu, h) := \sum_{\substack{m, n \in \mathbb{Z}, \\ |m| + |n| \geq 2}} t_\alpha(m, n; \mu, h) e^{i(mx + np)}$$

and

$$t_\alpha(m, n; \mu, h) := \frac{\lambda(m, n; \zeta_\alpha(\mu, h), h)}{a(\zeta_\alpha(\mu, h), h)}.$$

As previously noted, the symbols M_α and hence T_α are analytic with respect to x and p in the whole complex space. Furthermore, for real μ the solutions $\zeta_\alpha(\mu, h)$ are also real, hence, by (4.7), the coefficients $t_\alpha(m, n; \mu, h)$ are real as well and, as noted above, satisfy $t_\alpha(m, n; \mu, h) \equiv t_\alpha(-m, -n; \mu, h)$. Therefore, P_α takes only real values for real μ .

Now take an arbitrary $\varepsilon > 0$. By Lemmas 5.5 and 5.7 there exists $E_1 > E_0$ and $h_0 \in (0, h_*)$ such that

$$|t_\alpha(m, n; \mu, h)| \leq 2 \exp \left(-\sqrt{-\Re \zeta_\alpha(\mu, h)} (\sqrt{m^2 + n^2} - 1) \right).$$

for $\Re \zeta_\alpha(\mu, h) < -E_1$, $|\Im \zeta_\alpha(\mu, h)| < 4h_0$, $h \in (0, h_0)$, and $|m| + |n| > 1$. Then one can estimate for $|\Im x| + |\Im p| < \varepsilon^{-1}$ (hence for $|\Im x| < \varepsilon^{-1}$ and $|\Im p| < \varepsilon^{-1}$):

$$|T_\alpha(x, p; \mu, h)| \leq 2 \sum_{\substack{m, n \in \mathbb{Z}, \\ |m| + |n| \geq 2}} \exp \left(\frac{|m| + |n|}{\varepsilon} - \sqrt{-\Re \zeta_\alpha(\mu, h)} (\sqrt{m^2 + n^2} - 1) \right).$$

Therefore, there exists $E > E_1$ such that

$$|T_\alpha(x, p; \mu, h)| \leq \varepsilon \text{ for } |\Im x| + |\Im p| < \frac{1}{\varepsilon}$$

provided

$$|\mu| \leq 4, \quad \Re \zeta_\alpha(\mu, h) < -E, \quad |\Im \zeta_\alpha(\mu, h)| < 4h_0, \quad h \in (0, h_0).$$

Now using Lemma 6.2 we can choose $\alpha_0 \in (0, \alpha_1)$ such that $\Re \zeta_\alpha(\mu, h) < -E$ and $|\Im \zeta_\alpha(\mu, h)| < 4h_0$ for all $\alpha \in (0, \alpha_0)$, $h \in (0, h_0)$, and $|\mu| \leq 4$. This means that for all $\alpha \in (0, \alpha_0)$ and $h \in (0, h_0)$ the symbol $P_\alpha(\mu, h)$ is of strong type I with $\varepsilon(P_\alpha) \leq \varepsilon$. \square

Using Theorem 3.2, one can get the following corollary.

Corollary 6.5. *For any $E_0 > 0$ there exist $\alpha_0 > 0$ and $C > 0$ such that for $\alpha \in (0, \alpha_0)$ and sufficiently small irrational $(2\pi)^{-1}h$ admitting a continuous fraction expansion*

$$\frac{h}{2\pi} = \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \dots}}}$$

with integers n_j satisfying $n_j \geq C$, the spectrum of $H_{h, \alpha}$ in $(-\infty, -E_0)$ is a zero measure Cantor set.

Proof. For sufficiently small $\varepsilon(P_\alpha)$ and h as described, the set $\mu\text{-spec } \widehat{P}_{h,\alpha}(\mu)$ is a zero measure Cantor set in virtue of Theorem 3.2. In view of Theorem 6.4 it remains to emphasize that $z \mapsto m_\alpha(z, h)$ is an analytic topological isomorphism for sufficiently small α and h (see Lemma 6.3), hence the preimage of a zero-measure Cantor set is a zero-measure Cantor as well. \square

Remark 6.6. In [11] another solvable model was considered, namely, periodic quantum graphs with magnetic fields, and the expression for the spectrum obtained there in Theorem 13 is very close to the one of Theorem 6.4. Nevertheless, the essential difference was that the term T_α in the constructions analogous to Theorem 6.4 would just vanish, hence one had a reduction to the usual spectral problem for the discrete magnetic Laplacian. It is worthwhile to note that, even with such a simplification, the spectrum of non-isotropic quantum graphs was studied only numerically [21] up to now.

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